Journal of Engineering Mathematics **24**: 45–53, 1990.

Approximate Riemann solutions of the two-dimensional shallow-water equations

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Received 28 February 1989; accepted 22 May 1989

Abstract. A finite-difference scheme based on flux difference splitting is presented for the solution of the two-dimensional shallow-water equations of ideal fluid flow. A linearised problem, analogous to that of Riemann for gasdynamics, is defined and a scheme, based on numerical characteristic decomposition, is presented for obtaining approximate solutions to the linearised problem. The method of upwind differencing is used for the resulting scalar problems, together with a flux limiter for obtaining a second-order scheme which avoids non-physical, spurious oscillations. An extension to the two-dimensional equations with source terms, is included. The scheme is applied to a dam-break problem with cylindrical symmetry.

1. Introduction

The flow of water in a frictionless channel with rectangular cross section and smoothly varying bottom surface is governed by the two-dimensional shallow-water equations. The assumptions of hydrostatic pressure distribution and small bottom slope are used in deriving these equations [1]. Since analytical solutions of these equations are not generally available, they are solved numerically.

Several explicit and implicit finite-difference methods have been used to solve the shallow-water equations [2, 3, 4]. One feature of this set of hyperbolic equations is the formation of bores, i.e. discontinuous solutions, which can be difficult to represent accurately even if a shock-capturing method is used.

In the field of unsteady gasdynamics governed by the Euler equations, where shocks are frequently present, some authors have designed finite-difference schemes that have good shock-capturing properties, see e.g. [5]. These schemes solve linearised Riemann problems using upwind differencing and flux limiters to obtain shocks that are spread over two or three mesh points. An alternative approach to flux difference splitting was proposed by Vila [6] for the equations of isentropic gas dynamics and has been applied with success to the shallow-water equations [7]. The scheme of Godunov [8] solves Riemann problems exactly using an iterative procedure. Vila simplifies this iteration using approximate Riemann invariants, and achieves second-order accuracy by considering generalised Riemann problems, i.e. ones where the data is assumed to be piecewise linear discontinuous. In contrast, the scheme in [5] applies upwind differencing to a specially constructed set of scalar problems. Second-order accuracy is then achieved using classical second-order scalar schemes, limited to avoid non-physical oscillations in the solution.

In this paper a new scheme is presented for the shallow-water equations that incorporates the ideas mentioned earlier for the Euler equations. Although the derivation of this scheme is detailed, its implementation is straightforward. The resulting algorithm is efficient and produces satisfactory results for the two-dimensional test problem of a breaking dam.

2. Governing equations

The St. Venant equations governing the flow of water in a frictionless channel of rectangular cross section can be written in conservation form as

$$\mathbf{w}_t + \mathbf{F}_x + \mathbf{G}_z = \mathbf{f} + \mathbf{g}\,, \tag{2.1}$$

where

$$\mathbf{w} = (\phi,\ \phi u,\ \phi w)^T\,, \tag{2.2a}$$

$$\mathbf{F}(\mathbf{w}) = \left(\phi u,\ \phi u^2 + \frac{\phi^2}{2}\,,\ \phi uw\right)^T\,, \tag{2.2b}$$

$$\mathbf{G}(\mathbf{w}) = \left(\phi w,\ \phi uw,\ \phi w^2 + \frac{\phi^2}{2}\right)^T\,, \tag{2.2c}$$

$$\mathbf{f}(\mathbf{w}) = (0,\ g\phi h_x,\ 0)^T\,, \tag{2.3a}$$

$$\mathbf{g}(\mathbf{w}) = (0,\ 0,\ g\phi h_z)^T \tag{2.3b}$$

and

$$\phi = g(\eta + h)\,. \tag{2.4}$$

The quantities $\phi = \phi(x, z, t)$ and $u = u(x, z, t)$ and $w = w(x, z, t)$ represent g multiplied by the total height above the bottom of the channel and the components of the fluid velocity in the x and z directions, respectively, at a general position x, z and at time t. The gravitational constant is represented by g and the undisturbed depth of the water is given by $h(x, z)$. The elevation $\eta = \eta(x, z, t)$ above the plane $y = 0$ is measured in the vertical y direction. The special case when $h(x, z) = \text{constant}$ is considered first and the extension to the general case is developed from the special case. Equation (2.1) has been written so that the right-hand side does not contain any derivatives of flow variables. However, the vectors **f** and **g** are associated with derivatives in the x and z directions, respectively, as a consequence of the terms h_x and h_z. (N.B. Equations (2.1)–(2.4) represent conservation of mass and momentum. If we combine the mass and momentum equations we arrive at the more familiar equations of motion

$$u_t + uu_x + wu_z = -g\eta_x\,,$$

and

$$w_t + uw_x + ww_z = -g\eta_z\,.)$$

3. Operator splitting

We solve equation (2.1) using a Riemann solver together with the technique of operator splitting, [9], i.e. we solve successively

$$\mathbf{w}_t + \mathbf{F}_x = \mathbf{f} \tag{3.1a}$$

and

$$\mathbf{w}_t + \mathbf{G}_z = \mathbf{g} \tag{3.1b}$$

along x- and z-coordinate lines, respectively. We shall discuss the solution of equation (3.1a) and the solution of equation (3.1b) will then follow by symmetry.

4. Linearised Riemann problem

If the approximate solution of equations (3.1a) is sought along a coordinate line $z = z_0$ using a finite-difference method then the solution is known at a set of discrete mesh points $(x, z, t) = (x_j, z_0, t_n)$ at any time $t = t_n$. Following Godunov [8] the approximate solution $\mathbf{w}_j^n$ to $\mathbf{w}$ at (x_j, z_0, t_n) can be considered as a set of piecewise constants $\mathbf{w} = \mathbf{w}_j^n$ for $x \in (x_j - \Delta x/2,\ x_j + \Delta x/2)$ at time t_n where $\Delta x = x_j - x_{j-1}$ is a constant mesh spacing. A Riemann problem is now present at each interface $x_{j-1/2} = \frac{1}{2}(x_{j-1} + x_j)$ separating adjacent states $\mathbf{w}_{j-1}^n, \mathbf{w}_j^n$. If the shallow-water equations are linearised by considering the Jacobian matrix of the flux function $\mathbf{F}$ to be constant in each interval (x_{j-1}, x_j), the resulting equations can be solved approximately using explicit time stepping. The time step Δt is restricted so that the solutions of adjacent Riemann problems do not interact. The scalar problems that result from this analysis can be solved by upwind differencing; however, an approximate Jacobian matrix needs to be constructed in each interval so that shock-capturing is automatic.

5. Approximate Riemann solver

Consider firstly equations (3.1a) with $h(x, z) = \text{constant}$. The Jacobian matrix $A = \partial\mathbf{F}/\partial\mathbf{w}$ of the flux function $\mathbf{F}(\mathbf{w})$ has eigenvalues λ_i with corresponding eigenvectors $\mathbf{e}_i$, $i = 1, 2, 3$ given by

$$\lambda_1 = u + \sqrt{\phi}\,, \qquad \mathbf{e}_1 = (1, u + \sqrt{\phi}, w)^T\,, \tag{5.1a}$$

$$\lambda_2 = u - \sqrt{\phi}\,, \qquad \mathbf{e}_2 = (1, u - \sqrt{\phi}, w)^T\,, \tag{5.1b}$$

$$\lambda_3 = u\,, \qquad \mathbf{e}_3 = (0, 0, 1)^T\,. \tag{5.1c}$$

This information can be used to develop approximate solutions of the Riemann problem of Section 4.

Consider two adjacent states $\mathbf{w}_L, \mathbf{w}_R$ (left and right) given at either end of the cell (x_L, x_R) on an x-coordinate line $z = z_0$, and consider also the algebraic problem of finding an approximate Jacobian $\tilde{A} = \tilde{A}(\mathbf{w}_L, \mathbf{w}_R)$ in this cell such that

$$\tilde{A}\Delta\mathbf{w} = \Delta\mathbf{F}\,, \tag{5.2}$$

where $\Delta(\cdot) = (\cdot)_R - (\cdot)_L$. A solution to this problem, for arbitrary jumps $\Delta\mathbf{w}$, can be used to obtain a conservative scheme with good shock-capturing properties when incorporated with operator splitting. Equivalently, approximate eigenvalues $\tilde{\lambda}_i$ and corresponding eigenvectors $\tilde{\mathbf{e}}_i$ of $\tilde{A}$ can be sought such that

$$\Delta\mathbf{w} = \sum_{i=1}^{3} \tilde{\alpha}_i \tilde{\mathbf{e}}_i \tag{5.3a}$$

and

$$\Delta\mathbf{F} = \sum_{i=1}^{3} \tilde{\lambda}_i \tilde{\alpha}_i \tilde{\mathbf{e}}_i \,, \tag{5.3b}$$

where $\tilde{\alpha}_i$ are wavestrengths prescribed in terms of the arbitrary jump $\Delta\mathbf{w}$.

By initially considering small jumps $\Delta\mathbf{w}$, so that equations (5.3a,b) are satisfied to within $O(\Delta^2)$, a solution of equations (5.3a,b) can be determined [10]. The required approximate eigenvalues, eigenvectors and wavestrengths are

$$\tilde{\lambda}_1 = \tilde{u} + \tilde{\psi}\,, \qquad \tilde{\mathbf{e}}_1 = (1, \tilde{u} + \tilde{\psi}, \tilde{w})^T\,, \tag{5.4a}$$

$$\tilde{\lambda}_2 = \tilde{u} - \tilde{\psi}\,, \qquad \tilde{\mathbf{e}}_2 = (1, \tilde{u} - \tilde{\psi}, \tilde{w})^T\,, \tag{5.4b}$$

$$\tilde{\lambda}_3 = \tilde{u}\,, \qquad \tilde{\mathbf{e}}_3 = (0, 0, 1)^T\,. \tag{5.4c}$$

$$\tilde{\alpha}_1 = \frac{1}{2}\,\Delta\phi + \frac{1}{2}\,\frac{\tilde{\phi}}{\tilde{\psi}}\,\Delta u\,, \tag{5.5a}$$

$$\tilde{\alpha}_2 = \frac{1}{2}\,\Delta\phi - \frac{1}{2}\,\frac{\tilde{\phi}}{\tilde{\psi}}\,\Delta u\,, \tag{5.5b}$$

$$\tilde{\alpha}_3 = \tilde{\phi}\,\Delta w \tag{5.5c}$$

where the approximations to u, w, ϕ and $\sqrt{\phi}$ in (x_L, x_R) are given by

$$\tilde{S} = \frac{\sqrt{\phi_R}\,S_R + \sqrt{\phi_L}\,S_L}{\sqrt{\phi_R} + \sqrt{\phi_L}}\,, \qquad S = u \text{ or } w\,, \tag{5.6a,b}$$

$$\tilde{\phi} = \sqrt{\phi_R \phi_L} \tag{5.6c}$$

and

$$\tilde{\psi} = \sqrt{\tfrac{1}{2}(\phi_R + \phi_L)}\,. \tag{5.6d}$$

respectively. Thus, using equation (5.3b), equation (3.1a) with $h(x, z) = \text{constant}$ can be approximated by

$$\frac{\mathbf{w}_P^{n+1} - \mathbf{w}_P^n}{\Delta t} + \frac{\sum_{i=1}^{3} \tilde{\lambda}_i \tilde{\alpha}_i \tilde{\mathbf{e}}_i}{\Delta x} = \mathbf{0}\,, \tag{5.7}$$

along an x-coordinate line, $z = z_0$, where Δx and Δt represent the mesh spacing in the x and t directions, respectively, and the point P may be L or R. Upwind differencing now applied to equation (5.7) gives the following first-order algorithm for the solution of equation (3.1a) with $h(x, z) = \text{constant}$:

$$\text{add } -\frac{\Delta t}{\Delta x}\tilde{\lambda}_i\tilde{\alpha}_i\tilde{\mathbf{e}}_i \quad \text{to } \mathbf{w}_R \text{ when } \tilde{\lambda}_i > 0$$

or (5.8)

$$\text{add } -\frac{\Delta t}{\Delta x}\tilde{\lambda}_i\tilde{\alpha}_i\tilde{\mathbf{e}}_i \quad \text{to } \mathbf{w}_L \text{ when } \tilde{\lambda}_i < 0\,.$$

Thus, we note the direction of flow of information given by the approximate eigenvalues $\tilde{\lambda}_i$ and use this information to update the solution consistent with the theory of characteristics of equation (3.1a). In addition, second-order transfers of these first-order increments can be made to achieve higher accuracy, providing they are limited to maintain monotonicity [11]. The use of these 'flux-limiters' improves accuracy without introducing non-physical spurious oscillations, especially at bores. To allow depression waves to be treated correctly, the first-order increment can be considered as two separate increments being sent to either end of the cell [12].

6. Extensions

Consider now the inhomogeneous equation (3.1a) where $h(x, z)$ is smoothly varying and a 'source-term' $f = (0, g\phi h_x, 0)^T$ is present. This term, however, contains no derivatives of flow variables, and therefore the scheme of Section 5 can be retained for shock-capturing. Glaister has demonstrated [13] that for linearised systems the source term $\mathbf{f}$ should be upwinded in the same way as $\mathbf{F}_x$. This has been used successfully for axially symmetric, compressible flows [14]. Specifically, approximating $\mathbf{f}$ in the interval (x_L, x_R) by $\tilde{\mathbf{f}} = (0, g\tilde{\phi}\,\Delta h/\Delta x, 0)^T$, where $\tilde{\phi} = \sqrt{\phi_R\phi_L}$, and projecting

$$\tilde{\mathbf{f}} = -\frac{1}{\Delta x}\sum_{i=1}^{3}\tilde{\lambda}_i\tilde{\beta}_i\tilde{\mathbf{e}}_i\,, \tag{6.1}$$

enables equations (3.1a) to be solved approximately. (N.B. $\Delta h = h(x_R, z_0) - h(x_L, z_0)$, where $z = z_0$ is the x-coordinate line being considered). The first-order algorithm can be written as in equation (5.8) where the $\tilde{\alpha}_i$ are replaced by modified wavestrengths $\tilde{\gamma}_i = \tilde{\alpha}_i + \tilde{\beta}_i$. Solving equation (6.1) gives

$$\tilde{\beta}_{1,2} = \mp\frac{g\tilde{\phi}\,\Delta h}{2\tilde{\psi}(\tilde{u} \pm \tilde{\psi})}\,, \qquad \tilde{\beta}_3 = 0\,. \tag{6.2a,c}$$

7. A test problem

The problem of a breaking circular dam can be used to test the scheme of Section 5. Consider a channel, $x \geqslant 0$, $z \geqslant 0$, with rigid walls along $x = 0$ and $z = 0$, whose bottom surface is flat, and a barrier placed along $\sqrt{x^2 + z^2} = 0.5$. The water on one side of the barrier is at a different height to that on the other. At time $t = 0$ the barrier is removed and the resulting flow consists of a bore travelling towards $x = z = 0$ and a depression wave travelling out from the walls. To treat this problem numerically consider a fixed region $0 \leqslant x$, $z \leqslant 1$ with a barrier at $\sqrt{x^2 + z^2} = 0.5$. In $\sqrt{x^2 + z^2} > 0.5$ the water height is determined by ϕ_1 and on the other side by $\phi_0 < \phi_1$. This problem deals with ideal fluid flow and does not take into account real flow effects such as wall shear. The exact solution is a bore travelling towards $x = z = 0$, which is subsequently reflected from the origin. (This problem is part of the general problem in the x-z plane with a circular barrier along $\sqrt{x^2 + z^2} = 0.5$.)

8. Numerical results

Numerical results are given for the dam-break problem of Section 7 using the finite-difference scheme of Section 5. We take $\phi_0 = 1$, $\phi_1 = 2$ and the 'Minmod' limiter [11] has been used so that the resulting scheme is second-order accurate, but no spurious oscillations are produced. The results given in Figs 1–4 represent 31 equally spaced elevation contours at times $t = 0.12$, 0.24, 0.48 and 0.60 respectively, and 50×50 mesh points have been used. We see the propagation of the bore towards the origin and its subsequent reflection. We also plot the solution along the line $x = y$.

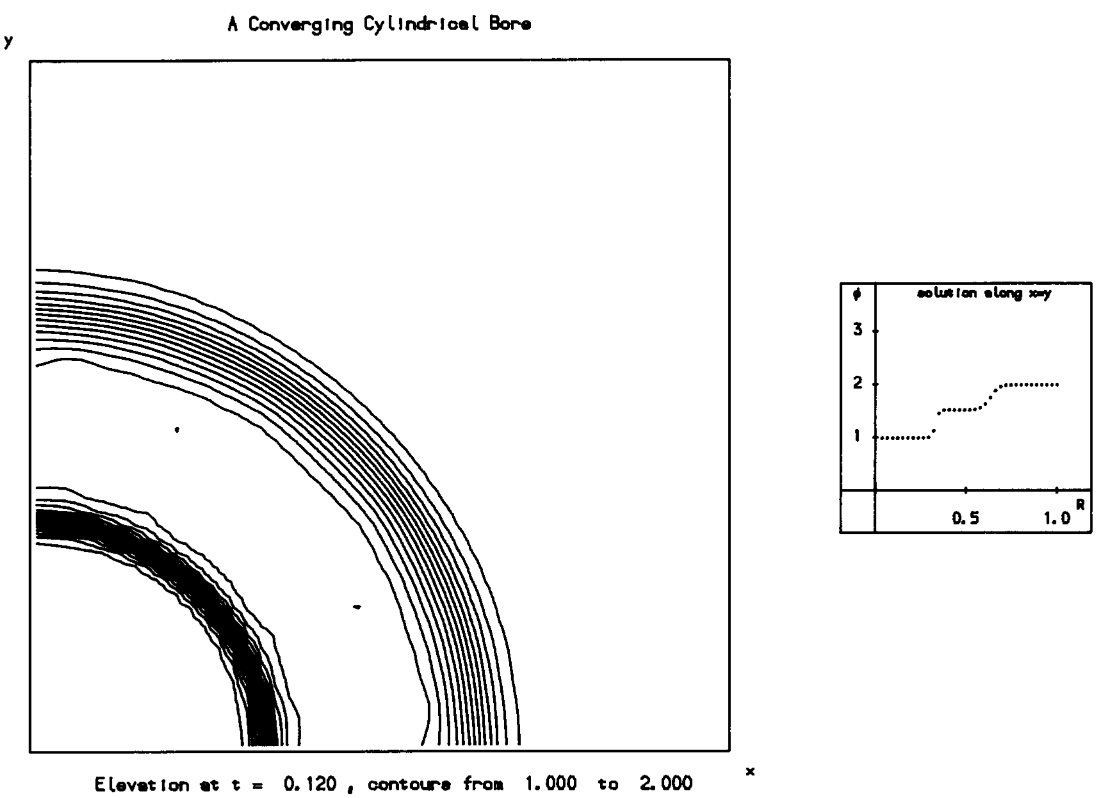

Fig. 1. Solution of the dam-break problem at $t = 0.12$.

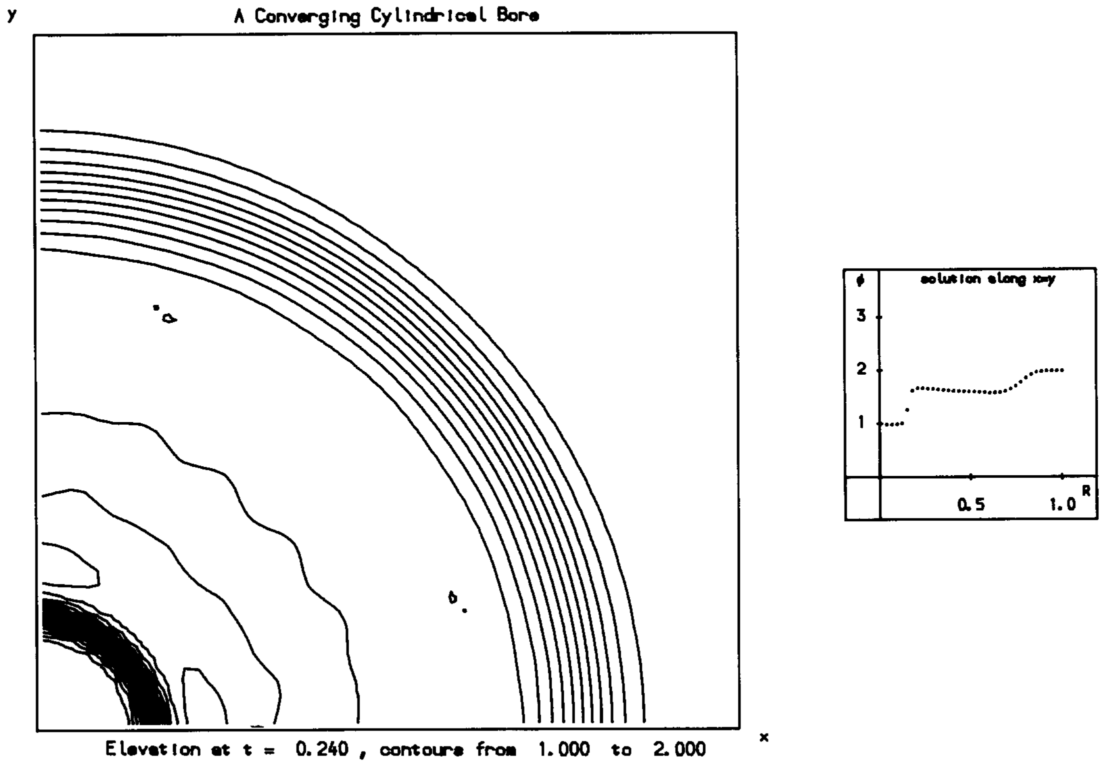

Fig. 2. Solution of the dam-break problem at $t = 0.24$.

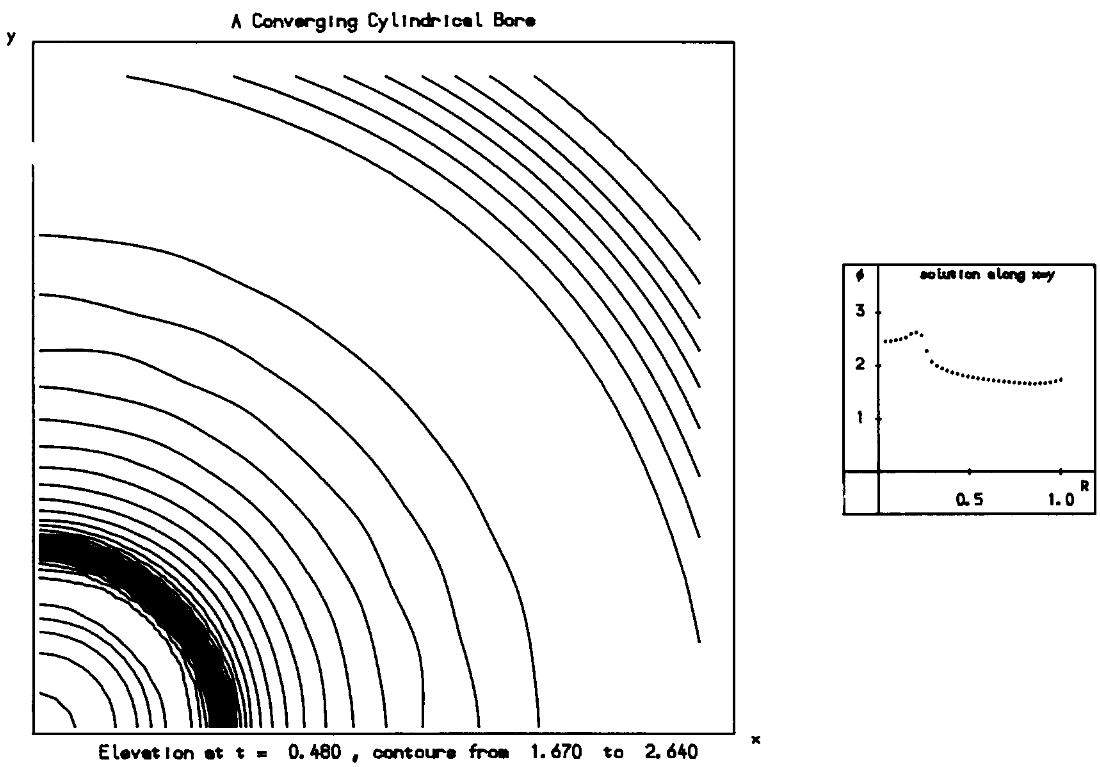

Fig. 3. Solution of the dam-break problem at $t = 0.48$.

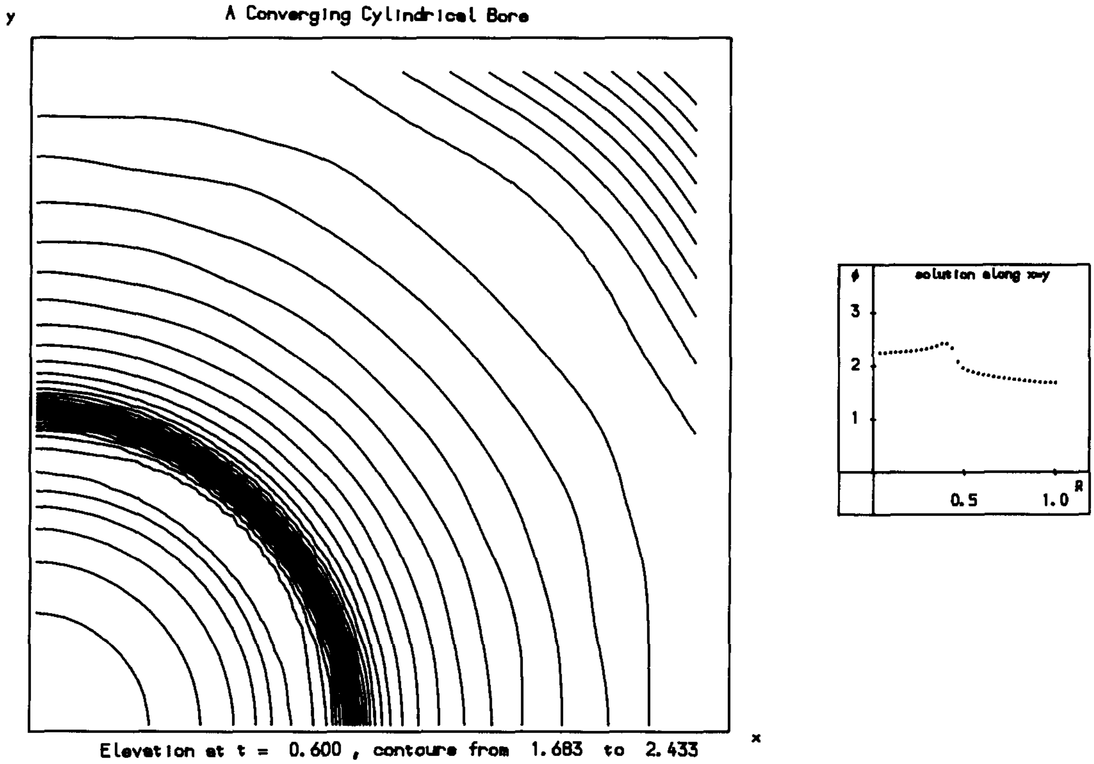

Fig. 4. Solution of the dam-break problem at $t = 0.60$.

The explicit finite-difference scheme of Section 5 is computationally efficient and the c.p.u. time taken to compute the results given above was as follows. Using an Amdahl V7 with 50×50 mesh points and the 'Minmod' limiter takes 0.75 c.p.u. seconds to compute one time step and a total of 15 c.p.u. seconds to reach a real time of 0.06 using 20 time steps.

9. Conclusions

A conservative finite-difference scheme is presented for the solution of the two-dimensional shallow-water equations based on flux difference splitting. By considering linearised Riemann problems, and solving these approximately using upwind differencing, enables the flow resulting from a dam-break to be predicted satisfactorily. The resulting scheme is computationally efficient and can be used with confidence to predict accurate solutions to two-dimensional flows.

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